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Application of algebraic topology to fingerprint recognition

Aplicación de la topología algebraica al reconocimiento de impresiones dactilares.

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Abstract

In the present work, an algorithm for fingerprints verification based on an application of the algebraic topology is presented. Specifically, we propose a representation of impressions as simplicial complexes and the definition of local structures based on local filtrations ordering from the complexes. This filtrations are determined by neighboring minutiae. It is also proposed the extraction of a set of features based on the analysis of the homology variation in these filtrations. The features combine information about the quantity and connectivity of papillary ridges in the local structures. In addition, a matching method based on the extracted topological information is presented. This paper shows that this information is discriminative and can be used in combination with classic geometric features to improve the description of local structures of the impressions and the accuracy in the comparison.

Keywords: algebraic topology, homology, fingerprints, topology, verification

Resumen

En este trabajo se presenta un algoritmo para la verificación de impresiones dactilares basado en una aplicación de la topología algebraica. Específicamente, se propone una representación de las impresiones como complejos simpliciales y la definición de estructuras locales de las impresiones a partir de filtraciones ordenadas locales de los complejos. Estas filtraciones quedan determinadas por minucias vecinas. También se propone la extracción de un conjunto de rasgos basados en el análisis de la variación de la homología en estas filtraciones. Estos rasgos combinan información acerca de la cantidad y la conectividad de las crestas papilares en las estructuras locales definidas. Además, se presenta un método de cotejo de las impresiones basado en la información topológica extraída. En este trabajo se muestra que esta información es discriminativa y puede usarse en combinación con rasgos geométricos clásicos para mejorar la descripción de estructuras locales de las impresiones y la eficacia en la comparación.

Palabras claves: homología, impresiones dactilares, topología, topología algebraica, verificación

Introduction

Fingerprint is one of the most popular and used biometric feature due to its uniqueness and immutability [Peralta et al. \(2015\)](#). Fingerprint verification is the process of deciding if two impressions belong to the same finger or not. A lot of methods have been proposed in the literature for fingerprint verification with excellent results [Peralta et al. \(2015\)](#). However it is not a solved problem. When the fingerprint image quality is poor, the nonlinear distortion is high or the overlap region is too small, the problem become very hard [Jain et al. \(2010\)](#). A constantly objective of researchers is to obtain features that allow more robust matching.

Topology is a mathematical field that studies the properties of the bodies that remain unchanged by continuous transformations. It usually focuses only on features such as shape, size and position; although it is also possible to integrate the objects geometry. Algebraic topology is a branch of topology based on the abstract algebra for the study of topological spaces. Algebraic topology can study structural relationships of spaces that are usually ignored by most common geometric features, for example the space connectivity. These information has been applied in some fields of pattern recognition such as human gait recognition [Lamar et al. \(2013\)](#) with good results. In fingerprint recognition many algorithms uses geometrical information extracted from local regions of the impression for matching. The topological information can be used to improve the description of local regions where non linear distortions can cause significant changes in the geometry but not in the topology of ridge patterns. Also, it can be used as a new feature when the overlap region is small and few geometrical information is available. The topological information can be combined with classical features to obtain a more accurate matching.

In this work we propose a representation of fingerprints as topological spaces through simplicial complexes. Then, we propose a set of features vectors extracted from the analysis of the homology of locals regions in the simplicial complex. At the end, we propose an algorithm to match two fingerprints based on these vectors and an analysis of its accuracy.

Related Work

Methods based in minutiae are the most well-known and used for fingerprint recognition [Maltoni et al. \(2009\)](#). Each minutia may be described by a set of attributes like its position, orientation and type, among others. Usually, they are consider as a triplet $m = \{x, y, \theta\}$ (x, y - minutia location coordinates, θ - minutia ridge orientation). The minutia based methods can be grouped into two families: Global minutiae matching and Local minutiae matching [Peralta et al. \(2015\)](#).

In global matching, usually the fingerprints are represented as a set of minutiae $T = \{m_1, m_2, \dots, m_n\}$. Given

two representations T and T' the final goal is to perform a match between the sets and give a similarity dependent of the number of minutiae matches. These methods obtain high distinctiveness because they capture fingerprint global spatial relationships [Maltoni et al. \(2009\)](#). On the other hand, generally they have a high computational cost because they need to perform an important alignment process to fit the translation and rotation differences. Examples of algorithms based in global matching are [Chen et al. \(2013\)](#), [Liu et al. \(2004\)](#).

Local matching consists of comparing two fingerprints according to local minutiae structures [Maltoni et al. \(2009\)](#). These structures are formed by attributes extracted from a minutia neighborhood and the relationship with respect to closer minutiae. The structures are invariant to global transformations (translation and rotation). Compared to global matching, the local technique present low computational complexity and high distortion tolerance. It also allow to perform matching with partial information [Peralta et al. \(2015\)](#). However the local matching presents low distinctiveness because they do not take into account the global relationships. Actually, the better results can be obtained by implementing hybrid strategies that perform a local matching to robustly determine pairs of minutiae as alignment candidates followed by a consolidation stage based in global matching [Maltoni et al. \(2009\)](#). The fingerprint local matching methods can be classified in several groups given its local structure representation similarities [Maltoni et al. \(2009\)](#), [Peralta et al. \(2015\)](#). Some of the most important are:

- Nearest Neighbors [Jiang and Yau \(2000\)](#), [Chikkerur et al. \(2006\)](#): This classification involve the firsts and classical works. The local structures are formed by one central minutia and the information about the relationship with respect to some other neighbor minutiae, usually the closests ones. Some of the common features extracted are the distances, angle differences and ridge count between the central minutia and it nearest-neighbor. These information serves as base to the subsequent works which improve these structure with other features.
- Fixed Radius [Ratha et al. \(2000\)](#): A fixed distance (d) and a circle of radius d around a central minutia are defined as the neighborhood of the minutia to be analyzed. The main difference with nearest neighbors is that the selection of the minutiae depends on the distance d . The principal problem in these methods are the minutiae mismatch in the region border due to the local distortions. The evolution of these methods is in the direction to avoid this issue.
- Minutiae Triangle [Tan and Bhamu \(2003\)](#), [Xu et al. \(2007\)](#): Triangles are constructed using minutiae as it vertices. Information about the triangles are incorporated such as they angles, side distance, number of ridge along the sides, triangle type, triangle direction, minutiae density in a local area and others.
- Texture [Tico and Kuosmanen \(2003\)](#), [Feng \(2008\)](#): Information from minutiae is combined with other kind of information relative to the local fingerprint appearance such as ridge local orientation or ridge

frequency. Usually sampling points are uniformly distributed around a minutiae and are used to calculate this information.

The algebraic topology has been applied in the pattern recognition and biometric fields [Alonso et al. \(2015\)](#). To the best of our knowledge it has not been applied in fingerprint recognition.

Topology Background

In order to understand the method presented in this work we provide some concepts about algebraic topology. Here we present the main definitions but it is a large and dense topic. We suggest consulting more specialized literature such as [Edelsbrunner and Harer \(2010\)](#) for better understanding.

Definition 1. [Zomorodian \(2009\)](#) A k -simplex (σ) is the convex hull of $k + 1$ affinely independent points $S = \{v_0, v_1, \dots, v_k\}$. The points in S are the vertex of the simplex. The dimension of σ is k .

A k -simplex has an intuitive interpretation in R^n . It can be a point, segment, triangle, tetrahedron or other entity of higher dimension. The simplices of different dimension are related by the operator face (\prec) (See [Def 2](#)). For example, the faces of a segment are their points and the faces of a triangle are their sides. A simplicial complex is a set of simplices and their faces where the intersection between simplices can be only on their faces (See [Def 3](#)). A simplicial complex defines a topological space [Zomorodian \(2009\)](#).

Definition 2. [Zomorodian \(2009\)](#) Let σ be a k -simplex defined by $S = \{v_0, v_1, \dots, v_k\}$. A simplex τ defined by $T \subseteq S$ is a **face** of σ and has σ as a **coface**. The relationship is denoted as $\tau \prec \sigma$ and $\sigma \succ \tau$.

Definition 3. [Zomorodian \(2009\)](#) A **simplicial complex** K is a finite set of simplices such that:

1. If $\sigma \in K$ and $\tau \prec \sigma \Rightarrow \tau \in K$.
2. If $\sigma_0, \sigma_1 \in K \Rightarrow (\tau = \sigma_0 \cap \sigma_1 = \emptyset \vee \tau \prec \sigma_0, \sigma_1)$.

Definition 4. [Zomorodian \(2009\)](#) A **filtration ordering** of a simplicial complex K is a full ordering of its simplices, such that each prefix of the ordering is a simplicial complex.

Simplicial homology is a topological invariant defined over simplicial complexes. It studies the way in which a k -simplex is connected to a $(k - 1)$ -simplex and how this relationship affects the creation of holes in the k and $k - 1$ dimensions.

Definition 5. [Edelsbrunner and Harer \(2010\)](#) A d -chain is a formal sum of d -simplices in a simplicial complex. The sum of two d -chains a and b is their symmetric difference $a \oplus b = (a \cup b) - (a \cap b)$.

Definition 6. Edelsbrunner and Harer (2010) The set of all d -chain in a simplicial complex K and the operation \oplus form a abelian group and is denoted as $C_d(K)$.

Definition 7. Edelsbrunner and Harer (2010) Let K be a simplicial complex and $\sigma \in K$ a d -simplex, the boundary $\partial_d(\sigma)$ is the set of all faces of σ in the dimention $d - 1$. The border of a d -chain is the symmetric difference of its simplices borders.

The operator ∂_d sets a relationship between the chain groups of differents dimentions. This relationship allows the definition of the homology groups (See Edelsbrunner and Harer (2010)). These groups capture important features of the simplicial complexes such as the holes in each dimention.

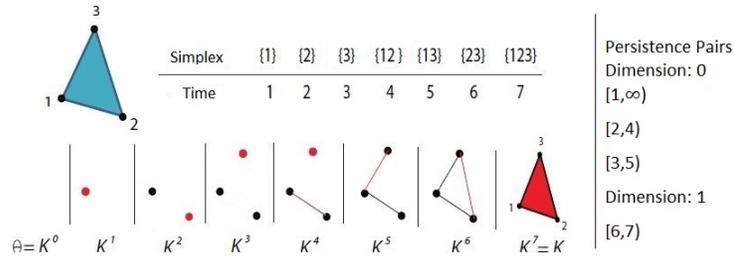


Figure 1. Filtration ordering from a simplicial complex represented by a triangle and its faces (left), and the persistence pairs resulting from for each dimention (right). Taken from Alonso et al. (2015)

Persistence homology studies how homology changes in a simplicial complex through a filtration ordering. It registers the moment in the filtration when a hole is created or destroyed for each dimension. The holes have an intuitive interpretation in each dimension, for example, in dimension 0 they are convex components, in dimension 1 they are cycles and in dimension 2 they are cavities. As a result of the persistence homology a list of pairs $[a, b)$ is obtained where a is the index of the d -simplex whose addition creates (born) a hole in the dimension d , and b is the index of the $(d + 1)$ -simplex whose addition causes that this hole disappear (die). If a hole does not disappear, it has a persistence pair $[a, \infty)$ (See Fig 1).

Proposed Method

We present our method split into two subsection: feature extraction and matching. For each subsection we use a set of definitions for a better explanation of our approach.

Feature extraction

The feature extraction stage is divided into four main steps. The first step is the representation of the fingerprint as a topological space through a simplicial complex (See Def 9). This complex is built from a skeleton image E of the fingerprint. A skeleton image is a binary image that is submitted to a thinning stage which allows for the ridge line thickness to be reduced to one pixel (See Fig 2). The simplicial complex was defined under the assumption that the major information of the fingerprint is determined by the ridge pattern configuration. The objective was to build a simplicial complex as representative to this pattern as possible.

Definition 8. An **edge set** $C(E)$ from a skeleton image E , is the set of all edges in the form $\langle (x, y), (u, v) \rangle$ where (x, y) and (u, v) are the coordinates of neighbor black pixels in E considering the 8-neighborhood of each pixel.

Definition 9. An **edge simplicial complex** of a skeleton image E , denoted as $S(E)$ is the set of all elements in $C(E)$ and its faces according the operator \prec (See Def 2).

The second step is the extraction of the filtrations ordering over the simplicial complex. It is the input of the homology persistence algorithm and is a crucial step because it defines the topological relationships that can be captured. Differing from Lamar et al. (2013), in the fingerprints case, it make no sense to create filtrations over all the complex due to the overlaped region in genuine impressions is not the same. For that reason, in this work we propose to make local filtrations in the simplicial complex. For defining the filtrations it is necessary to define some concepts:

Definition 10. The **center of a simplex** σ is the point which coordinates are equal to the mean of the simplex vertex coordinates. The **distance between a simplex and a minutia** $d(\sigma, m)$ is defined as the Euclidean distance between the simplex center and the minutia. The **distance between two simplex** $d(\sigma_i, \sigma_j)$ is defined as the Euclidean distance between their centers.

Definition 11. Let $S(E)$ be a simplicial complex defined over a skeleton imagen E and m_i a minutia in E ; it is defined as **neighborhood of radius r of m_i** to the set $\nu_r^i = \{\sigma_j \in S(E) | d(\sigma_j, m_i) \leq r\}$ where $r > 0$.

Definition 12. The **filtrations ordering** $g_k^+(m_i)$ and $g_k^-(m_i)$ are the ascending and descending ordering of ν_r^i elements respect to their distances $d(\sigma, m_i)$ where $r = d(m_i, m_j)$ and m_j is the k -th nearest minutia to m_i .

Definition 13. The set $G_k(E) = \{g_j^+(m_i), g_j^-(m_i) | \forall m_i \in M(E), 0 < j \leq k\}$ is the set of all filtration ordering of the impression E , centered in each minutia m_i . The value of k is called neighborhood size.

The filtrations of the definition 12 are an ordering of the simplices in a circular region centered in one minutia. The filtrations g^+ describe the ridge flow in the region growing from the center to the borders, and the g^- in

reverse (See Fig 1). These structures are rotation and translation invariant. An important consideration of these kind of filtrations is that the size of the filtration a discriminatory factor.

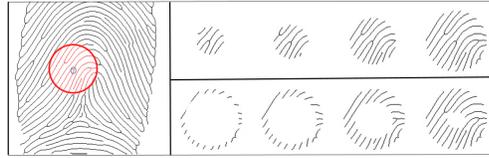


Figure 2. Filtrations $g_4^+(m_i)$ and $g_4^-(m_i)$ relative to a minutia m_i in a skeleton impression. In the left of the image the central minutia is drawn in blue and the region determined by its 4-neighbor in red. In the right are showed four moments in the filtrations $g_4^+(m_i)$ (up) and $g_4^-(m_i)$ (down).

The third step in the feature extraction stage is the analysis of the homology persistence. For each filtration in $G_k(E)$ their persistence intervals are calculated (See Def 14). For this objective was used an implementation of the algorithm known as sparse matrix reduction from Edelsbrunner and Harer (2010).

Definition 14. It is defined as l_{ijp}^+ and l_{ijp}^- to the pair lists resulting from homology persistence calculation over the filtrations ordering $g_j^+(m_i)$ y $g_j^-(m_i)$ respectively.

A list of pairs for each dimension in the simplicial complex result from the persistence calculation of one filtration. The index p in the term l_{ijp} represents the dimension of the list. In this work we propose to use only dimension 0 because the information in dimension 1 is very poor.

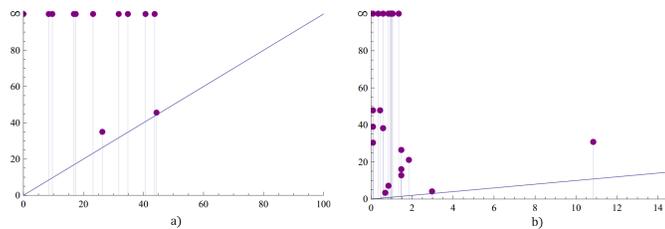


Figure 3. Persistence diagrams relative to the filtrations in Fig 1. In a) the diagram l_{i40}^+ relative to $g_4^+(m_i)$, in b) the diagram l_{i40}^- relative to $g_4^-(m_i)$. The x axis represent the born time and the y axis the death time. The blue line is the function $x = y$.

The persistence pairs can be represented in the persistence diagrams (See Fig 3). The interpretation of these diagrams is related to the connectivity history of the ridge flow through the filtration. For example, in the case of l_{i40}^+ (See Fig 3), many points appear with finite born time and infinite death time. This is because in this filtration generally each ridge appears as a convex component and continues in this way until the end. Some points with finite death time reflect the time when two ridges are joined and one component dies, for example,

in a bifurcation. In the case of l_{i40}^- many components appear for the first time because the ridges are cut by the circle border and they die when these ridges are joined through the filtration.

The complete set of l_{ij} lists of an impression E represent the topological information proposed in this work to extract from E . As a final step, we continue with the idea proposed in Lamar et al. (2013) and define a vector of numbers extracted from the diagrams (See Def 15).

Definition 15. Let l_{ij} be a list of persistence intervals and m the lower value such as $m \geq x$ and $m \geq y$, $\forall [x, y] \in l_{ij}$ with $y \neq \infty$, it is denoted as V_{ij} the vector of $2 * n$ integer values associated to l_{ij} with $n \leq m$ and is defined $\forall 0 \leq t < n$ as:

- $V_{ij}[2t] =$ number of intervals $[a, b] \in l_{ij}$ such as $a \leq \lfloor \frac{tm}{n} \rfloor$ and $b \geq \lfloor \frac{(t+1)m}{n} \rfloor$.
- $V_{ij}[2t + 1] =$ number of intervals $[a, b] \in l_{ij}$ such as $a > \lfloor \frac{tm}{n} \rfloor$ and $a \leq \lfloor \frac{(t+1)m}{n} \rfloor$.

The set of all V_{ij} is the final features of an impression for the matching process proposed in this work. The information captured by these vectors performs as a special ridge counter. The number of independent ridges that exists until the filtration moment $\lfloor \frac{(t+1)m}{n} \rfloor$ appears in the even positions of each vector, and the number of ridges that were born in the filtration interval $(\lfloor \frac{tm}{n} \rfloor, \lfloor \frac{(t+1)m}{n} \rfloor]$ appears in the odd positions.

Matching

For each minutia m_i in one impression E , a set of feature vectors are extracted, which describe a local region determined by m_i neighbor minutiae (See Def 16). In this work we propose a matching stage based in the comparison of these local regions (See Def 18). The similarity between two impressions is given by their p most similar regions. The best value of p can be estimated in a training stage. This value depend on the fingerprint overlap region and minutiae density.

Definition 16. The **topological descriptions from a minutia** m_i is defined as the set $b_i = \{V_{ij_0}, V_{ij_1} \dots V_{ij_k}\}$ of vectors associated to m_i . The set $B(E) = \{b_0, b_1 \dots b_n\}$ is defined as the set of all topological descriptions in an impression E .

Definition 17. The **similarity between two topological minutia descriptions** is defined as the function $S(b_i, b_j) = \frac{\sum_{k=0}^m d(V_{ik}, V_{jk})}{m}$ with $b_i, b_j \in B \wedge m = |b_i| = |b_j|$ and d is the Euclidean distance.

Definition 18. The **similarity between two impression** I and I' is defined by $Z(I, I') = \frac{\sum_{k=0}^p S_k}{p}$ where $\{S_0, S_1 \dots S_p\}$ are the p -th lowers values of $S(b_i, b'_j)$ with $b_i \in B(I) \wedge b'_j \in B(I')$ and the pairs (b_i, b'_j) are disjoint.

Global Matching

As was said before the global matching can improve the accuracy in the verification process. In this work we propose a global matching process inspired in [Jiang and Yau \(2000\)](#). The process consist on aligning the sets of minutiae $M(I)$ and $M(I')$ of two impression I and I' by the pair of central minutiae of the more similar regions selected in the local matching stage. If m_i is the minutia selected in one impression, the rest of the minutiae are rewritten in the form $m_j = \{d_{ij}, \theta_{ij}, \varphi_{ij}\}$ where d_{ij} is the distance between m_i and m_j , θ_{ij} is the angle differences between their local ridge direction and φ_{ij} is the angle between m_i local ridge direction and the line that joins m_i and m_j . Then the minutiae set is sorted by the d_{ij} parameter in an ascending way. Finally a simple matching process between the two sets is performed. The final score is $Z(I, I') = \frac{m}{p}$ where m is the number of minutia matched and $p = \frac{|M(I)|+|M(I')|}{2}$.

Experimental Results

To test our method we use the FVC2002 DB1 database [Maio et al. \(2002\)](#). This database was built with an optical sensor "TouchView II" by Identix with an image size of 388x374 (142 pixels) and a resolution of 500 dpi. The database contains 8 impression of 110 fingers split into two sets: DB1 (B) with 80 images to train and DB1 (A) with 800 images to test. We follow the same experimental protocol proposed in the competition. In these protocol, the principal measure used to compare the algorithms is the EER (equal error rate) but also are presented the FMR100, FMR1000, ZeroFMR (where FMR mean False Match Rate) [Maio et al. \(2002\)](#).

Our experiments were carried out in two directions. First, we tested our methods using as fingerprint similarity score only the result of the local matching. Secondly we analyze the accuracy of our method by adding the consolidation step based in the global minutia matching. Also, for each test, we changed the features vectors by other features vectors with some classical geometric information similar to the ones used by [Jiang and Yau \(2000\)](#) to compare the performance. These geometric feature vectors have the form $(d_{ij}, \theta_{ij}, \varphi_{ij})$ where these parameters have the same meaning as in the global matching (See: Proposed Method). This geometric information was also added to the topological vectors to analyse the behavior of the combined information.

Table 1 shows the best results obtained in the local matching for each type of features. We performed tests for integer values of parameters p and k in the range $0 < p \leq 10$, $0 < k \leq 10$ where p is the number of descriptions considered for the similarity (See Def 18) and k is the neighborhood size (See Def 13). Table 2 shows the best results obtained in the global matching (local matching + consolidation step) for each type of features. In this case, we used $p = 1$ (Because is selected the pair of minutiae descriptions that best match in the local comparison for align) and we consider the values $0 < k \leq 10$.

Table 1. Best results using local matching only. The best results were achieved in: case of T (topological features) for $p = 1$, $k = 5$, case of G (geometric features) for $p = 1$, $k = 2$ and case of T+G (topological + geometric features) for $p = 1$, $k = 2$.

Features	EER	FMR100	FMR1000	ZeroFMR	Average Time: Feature Extraction (s)	Average Time: Matching (s)
T	20.90	95.20	95.80	95.90	1.907	0.031
G	6.61	16.46	50.78	99.32	0.813	0.006
G+T	4.20	12.35	33.32	90.89	0.813	0.008

Table 2. Best results using local matching + consolidation. The best results were achieved in: case of T (topological features) for $k = 4$, case of G (geometric features) for $k = 2$ and case of T+G (topological + geometric features) $k = 2$.

Features	EER	FMR100	FMR1000	ZeroFMR	Average Time: Feature Extraction (s)	Average Time: Matching (s)
T	8.58	10.21	11.92	17.07	1.763	0.21
G	3.76	4.5	6.39	17.5	0.813	0.026
G+T	2.97	3.55	6.44	17.88	0.813	0.12

As shown in table 1, the local matching based on topological features alone does not offer good results. This is mainly because the local matching method is based on the selection of the most similar regions. In impostor impressions, is common to find many areas where the ridge pattern is very similar. Generally, these area, were selected by the matching method and these impostors impression received good evaluation results. Nevertheless, as shown in table 2, when we added the consolidation step, the EER was reduced from 20% to 8%. This is because the global spatial information helps to discriminate between impostors impressions. Also, it means that the selection of the most similar region for alignment in genuine impression, for alignment was correct in the majority of cases. This shows the discriminatory power of these features.

The result of $EER = 8,58\%$ is not one of the bests reported in the FVC2002 DB1 database. That means that these topological features by themselves are not enough for a completely verification algorithm. As was said in section Related Works and showed in table 1 and 2, the relationship between the minutiae geometrical features is very discriminative. What we aim to show with our work is that the combination of geometrical features with topological features may provide better results. This can be seen in row 3 of Table 2, where we achieved 2.97 of EER by the combination of both types of features, improving the EER of the geometrical features alone (row 2). This means that the topological information enriched the local region descriptions and allowed a better selection of alignment minutiae.

In the experiments analysis we find that topological information has better results in impressions where the minutia density is low. It makes sense because in these cases the minutiae neighborhood captures a bigger area

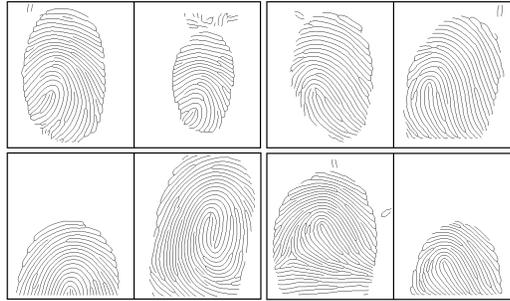


Figure 4. Pairs of genuine skeleton impressions from **FVC 2002 DB1A** that were correctly matched when the topological features were added in the local matching. In this impression can be observed a relative low minutiae density in the overlap region.

and a more complete description of the ridge pattern. Also, in some cases, when the overlap region is small and few minutiae exists, topological features allow a better matching (See Fig 4). The invariance to non linear distortions was not solved completely because the filtration size depend on minutiae neighbors, nevertheless the negative impact in the feature vectors by this concept is small. The main limitation of topological information is the noise in the ridge connectivity which causes differences in the convex components history.

Conclusions

In this work we presented an algorithm for fingerprint recognition based on the topological analysis of the ridge pattern through persistence homology. The proposed topological description works like a special ridge counter in the minutiae neighborhood. Experiments showed that this information is discriminative but not enough for an effective matching algorithm by themselves. However the topological information was used to improve the description of fingerprints local structures in combination with other geometrical features. This work is the first application of this topic to fingerprint recognition. In the future we may consider the representation of the fingerprint as a different simplicial complex or the definitions of other filtrations that capture a different information. Also, it is possible to extend this idea to palm print recognition.

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